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Limit Theorem for the Hit Time of Mappings of a Circle with Break

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Abstract: In this paper, it is proved the limit theorem for distribution functions $\Phi_n^{(k)}(t)$, $k \geq 2$ of $k-$th entrance times in $V_n(x_b)$.

Keywords: homeomorphisms of a circle, hit time, rotation number.

The theory of homeomorphisms of the circle is one of the important directions of modern theory of nonlinear systems. Its creation is connected mainly with the names of outstanding mathematicians A. Poincare, A. Danjoy, A.N.Kolmogorov, V.I.Arnold, Yu.Moser, M.R.Herman, J.C.Yoccoz, Ya.G.Sinai and D.Ornstein. Circle homeomorphisms are also important for their applications in the natural sciences (see [1], [2], [3], [4], [5], [6], [7], [8]). For the first time homeomorphisms of the circle were studied in the classical work of A. Poincare [8] in connection with problems of celestial mechanics.

Any orientation-preserving homeomorphisms of the unit circle $S^1 = R^1 / Z^1 \cong [0,1)$ is given by the formula $T_f x = f(x) \,(\mathrm{mod}1)$, $x \in S^1$, here $f(x)$- is a continuous, strictly increasing function on R^1, that satisfies the condition $f(x+1) = f(x)+1$, $x \in R^1$. The function f is called the defining function or the lifting of the homeomorphisms T_f. Note that the lift8 is defined up to an additive integer constant, but this ambiguity is eliminated by the initial condition $0 \leq f(0) < 1$. A. Poincare showed [2] that for any $x \in R^1$ there is a finite limit $\lim\limits_{n\to\infty} \dfrac{f^{(n)}(x)}{n} = \rho_f$, here and everywhere below $f^{(n)}(x)$- denotes the n-th iteration of the function $f(x)$. The number $\rho = \rho_f$ called the rotation number, does not depend on the choice of x and is the most important numerical characteristic of the homeomorphisms T_f. Данжуа показал [7], that if $f \in C^1(R^1), \operatorname*{var}\limits_{S^1} \ln f'(x) < \infty$ and the rotation number $\rho = \rho_f -$ are irrational, then there exists a circle homeomorphism T_φ such that $T_\varphi \circ T_f = T_\rho \circ T_\varphi$, where T_ρ, $x \in S^1$ is a linear rotation through an angle of ρ_f. The mapping T_φ is called a conjugation or a conjugating homeomorphisms.

It is well known that a homeomorphisms T_f with an irrational number rotation is strictly ergodic, i.e. has a unique probability invariant measure μ_f. A remarkable fact is that the conjugation T_φ and the invariant measure μ_f are related by the relation $T_\varphi x = \mu([x_0, x]), x \in S^1$. The last relation shows that the invariant measure μ_f is absolutely

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continuous if and only if φ is an absolutely continuous function. Fundamental results in the problem of conjugation smoothness were obtained in the works of V.I. Arnold [1], Yu. M. Khanin [8], J. Katznelson and D. Ornstein (see [7], [8]) and others.

The class of circle homeomorphisms with break-type singularities has been relatively little studied.

Definition 1. A point $x_b \in S^1$ is called a breaking point of a homeomorphisms T_f, if there are finite, positive, one-sided derivatives $f'(x_b \pm 0)$ and $\frac{f'(x_b - 0)}{f'(x_b + 0)} = \sigma_f(x_b) \neq 1$. The number $\sigma_f(x_b)$ is called the break value of the homeomorphism T_f at the point x_b.

Everywhere below we will assume that the rotation number $\rho = \rho_f$ is irrational. Let the continued fraction expansion of ρ look like: $\rho = [k_1, k_2, ..., k_n, ...]$, $k_n \geq 1$. Let's denote $\frac{p_n}{q_n} = [k_1, k_2, ..., k_n]$, $n \geq 1$. The numbers q_n - are called the times of the first return and satisfy the difference equation: $q_{n+1} = k_{n+1} q_n + q_{n-1}$, $n \geq 1$, $q_0 = 1$, $q_1 = k_1$.

Let $x_0 \in S^1$. Let's put $x_i = T_f^i x_0$, $i \geq 1$. Note that if n is odd, x_{q_n} is to the left of x_0, and if n- is even, to the right. Denote by $V_n(x_0)$ a closed segment connecting the points x_{q_n} and $x_{q_{n+1}}$. $V_n(x_0)$- is called the n- th renormalization neighborhood of point x_0. We define the Poincare mapping $\pi_n : V_n(x_0) \to V_n(x_0)$:

$$\pi_n(x) = \begin{cases} T_f^{q_{n+1}} x, & if \quad x \in [x_{q_n}, x_0); \\ T_f^{q_n} x, & if \quad x \in [x_0, x_{q_{n+1}}]. \end{cases}$$

According to the general scheme of the RG method, we are mainly interested in the behavior of the Poincare mapping $\pi_n(x)$ as $n \to \infty$. Since the length of the segment $V_n(x_0)$ tends exponentially to zero and $q_n \to \infty$ as $n \to \infty$, it is convenient to study the behavior of поведение $\pi_n(x)$ in the new renormalized coordinates. We introduce the renormalized coordinates z to $V_n(x_0)$: $x = x_0 + z(x_0 - x_{q_n})$. This shows that in the new coordinates $x_0 \to 0$, $x_{q_n} \to -1$. Let us denote by a_n and $(-b_n)$ the renormalized coordinates of points $x_{q_{n+1}}$ and $x_{q_n + q_{n+1}}$ respectively, i.e. $a_n = \frac{x_{q_{n+1}} - x_0}{x_0 - x_{q_n}}$, $b_n = \frac{x_0 - x_{q_n + q_{n+1}}}{x_0 - x_{q_n}}$. In new coordinates, mapping $\pi_n(x)$ corresponds to the following pair (f_n, g_n):

$$f_n(z) = \frac{f^{q_{n+1}}(x_0 + z(x_0 - x_{q_n})) - x_0 - p_{n+1}}{x_0 - x_{q_n}}, \qquad z \in [-1, 0],$$

$$g_n(z) = \frac{f^{q_n}(x_0 + z(x_0 - x_{q_n})) - x_0 - p_n}{x_0 - x_{q_n}}, \qquad z \in [0, a_n],$$

Sinai and Khanin showed that for sufficiently smooth diffeomorphisms functions $f_n(z)$ and $g_n(z)$ are close to linear functions.

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We define the following linear-fractional functions

$$F_n(z)=\frac{a_n+(a_n+b_n m_n)z}{1+(1-m_n)z},\quad G_n(z)=\frac{-a_n c_n+(c_n-b_n m_n)z}{a_n c_n+(m_n-c_n)z},$$

where $c_n=\sigma^{(-1)^n}$ and $m_n=\exp\left\{\sum_{i=0}^{q_{n+1}-1}\int_{x_i+q_n}^{x_i}\frac{f''(y)}{2f'(y)}dy\right\}$.

It is well known that the transformation of the renormalization group in the set of circle homeomorphisms with breaks has periodic trajectories. Denote by X the set of pairs of strictly increasing functions $(f(x),\ x\in[-1,0];\ g(x),\ x\in[0,\alpha])$ satisfying the following conditions:

$a)\quad f(0)=\alpha,\ g(0)=-1,\quad f(-1)=g(\alpha),\quad f(-1)<0,\quad f^{(2)}(-1)\ge 0;$

$b)\quad f(x)\in C^{2+\varepsilon}([-1,0]),\ g(x)\in C^{2+\varepsilon}([0,\alpha])$, for some $\varepsilon>0$.

Let us define the transformation of the renormalization group $R_b: X\to X$:

$$R_b(f(x),x\in[-1,0];\ g(x),\ x\in[0,\alpha])=(\tilde f(x),x\in[-1,0];\ \tilde g(x),\ x\in[0,\alpha'])\ ,$$

where $\tilde f(x)=-\alpha^{-1}f(g(-\alpha x)),\ \tilde g(x)=-\alpha^{-1}f(-\alpha x),\ \alpha'=-\alpha^{-1}f(-1)$.

Let's put $c=f'(-0)\cdot\left(g'(+0)\right)^{-1}$, i.e. c - the value of the break of the pair (f,g) at the point $x=0$. In the work of Wool and Khanin, it is proved that for a fixed c and the rotation number equal to the “golden ratio”, the transformation R_b has a unique periodic orbit $\{f_i(x),\ g_i(x),\ i=1,2\}$ of period two. Functions $f_i(x,c_i),\ g_i(x,c_i),\ i=1,2$ are:

$$f_i(x)=\frac{(\alpha_i+c_i x)\beta_i}{\beta_i+(\beta_i+\alpha_i-c_i)x},\ g_i(x)=\frac{\alpha_i\beta_i(x-c_i)}{\alpha_i\beta_i c_i+(c_i-\alpha_i-c_i\beta_i)x}\ ,$$

where $\alpha_1=\dfrac{c_1-\beta_0^2}{1+\beta_0},\ \ \alpha_2=\dfrac{c_2-\beta_0^2}{1+\beta_0},\ c_1=c,\ c_2=c^{-1},\ \beta_1=\beta_2=\beta_0$, and the number β_0- is the only root of the equation $\beta^4-\beta^3-\beta^2\dfrac{(c+1)^2}{c}-\beta+1=0$ belonging to the interval $(0,1)$. Using pairs $(f_i,\ g_i),\ i=1,2$, we define circle homeomorphisms T_i: $T_i(x)=l_i\left(f_i(l_i^{-1}(x))\right)$, if $0\le x<\left(1+\alpha_i\right)^{-1}$ and $T_i(x)=l_i\left(g_i(l_i^{-1}(x))\right)$, if $\left(1+\alpha_i\right)^{-1}\le x<1$. The rotation numbers of these homeomorphisms are equal to the “golden section”. We will study the homeomorphism T_1 . The homeomorphism T_2 is studied in a similar way. The homeomorphism T_1 will be re-denoted by T_b . Denote by $B(T_b)$ the set of all C^1- homeomorphisms of the circle conjugate to T_b.

Theorem 1. For all mappings $\boldsymbol{T\in E(T_b)}$ there is a unique continuous (in the Tikhonov topology) function $\boldsymbol{U: I_+\to R^1}$ with the following properties:

1. For any $\vec\varepsilon=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k,\varepsilon_{k+1},\ldots\varepsilon_n,\ldots)$ and

$\vec b=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k,b_{k+1},\ldots b_n,\ldots)$ out of space I_+ correct

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$$\left|U(\vec{\varepsilon})-U(\vec{b})\right| \le K_1 \alpha^k$$

where $\alpha = \alpha(T_b) \in (0,1)$ and constant $K_1 > 0$ does not depend on $\vec{\varepsilon}, \vec{b}$ and k.

2. Let $1 \le r < n,\ \Delta(a_1, a_2, \ldots, a_n) \in \xi_n(x_0, T),$

$\Delta(a_1, a_2, \ldots, a_r) \in \xi_r(x_0, T)$, $\Delta(a_1, a_2, \ldots, a_n) \subset \Delta(a_1, a_2, \ldots, a_r)$., .

Then

$$l(\Delta(a_1, a_2, \ldots, a_n)) = l(\Delta(a_1, a_2, \ldots, a_r)) \times$$
$$\times (1 + \psi_1(a_1, a_2, \ldots, a_n); T) \times$$
$$\times \exp\{ \sum_{s=r+1}^{n} U(a_s, a_{s-1}, \ldots, a_r, \ldots, a_1, \vec{\gamma}(a_1))\}$$

where $\left|\psi_1(a_1, a_2, \ldots, a_n; T)\right| \le const\, \alpha^r$.

Let $n \ge 1$ and $V_n(x_0)$ - $n-$ be the renormalization neighborhood of the point $x_0 \in S^1$.

Let's define $E_n^{(1)}(x) = \min\{i \ge 1 : T_f^i x \in V_n(x_0)\}$,

$E_n^{(k)}(x) = \min\{i \ge E_n^{(k-1)}(x) : T_f^i x \in V_n(x_0)\}, k \ge 1.$ Consider random variables $D_n^{(k)}(x) = E_n^{(k)}(x) - E_n^{(k-1)}(x)$. Note that $D_n^{(1)}(x) = E_n^{(1)}(x)$ takes values from 1 to q_{n+1} , while $D_n^{(k)}(x)$ takes only two values: q_n and q_{n+1} . Let us introduce normalized random variables: $\overline{D}_n^{(k)}(x) = q_{n+1}^{-1}\, D_n^{(k)}(x)$. The problem is to study the convergence of the distribution function for random variables $\overline{D}_n^{(k)}(x)$ as $n \to \infty$, as well as their limiting distributions.

Denote $F_n^{(k)}(t) = \mu_f(\{x \in S^1 : \overline{D}_n^{(k)}(x) \le t\}), t \in R^1$. Note that the functions $F_n^{(k)}(t)$ coincide with the corresponding distribution functions for linear rotation T_ρ . De Faria and Coelo proved that, depending on the rotation number ρ, the limit distribution of the convergent subsequence $\{F_{n_i}^{(1)}(t)\}$ is either uniform or continuous and piecewise linear on the interval $[0,1]$. And in case $k > 1$ the limit distribution for a convergent subsequence $\{F_{n_i}^{(k)}(t)\}$ is either the distribution of a random variable $X \equiv 1$ or a stepwise distribution with two discontinuity points.

Let us denote by $\Phi_n^{(k)}(t)$ the distribution function of $\overline{D}_n^{(k)}(x)$ with respect to the Lebesgue measure l: $\Phi_n^{(k)}(t) = l(\{x \in S^1 : \overline{D}_n^{(k)}(x) \le t\}),\ t \in R^1$. If a diffeomorphism T_f is smoothly conjugate to a linear rotation T_ρ , then for the sequence $\{\Phi_n^{(k)}(t)\}$ all the above statements related to $\{F_n^{(k)}(t)\}$ are also valid. On the other hand, for circle homeomorphisms with one breakpoint (or with several breakpoints lying on the same orbit and with a non-trivial product of breakpoints) and with an irrational rotation number ρ_f, the conjugating homeomorphism T_φ is

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singular. Take an arbitrary circle homeomorphism $T \in B(T_0)$. Recall that $E_n^{(k)}(x)$ means the time $k-$ of the point $x \in S^1$ hitting the $n-$th renormalization segment V_n. Let's denote $D_n^{(k)}(x) = E_n^{(k)}(x) - E_n^{(k-1)}(x),\ x \in S^1$. The random variable $D_n^{(k)}(x)$ takes only two values: q_n or q_{n+1}. We normalize it dividing by q_{n+1}^{-1}:

$$\overline{D}_n^{(k)}(x) = q_{n+1}^{-1} D_n^{(k)}(x) \ .$$

Let us denote by $\Phi_n^{(k)}(x)$ the distribution function of the random variable 94 $\overline{D}_n^{(k)}(x)$ with respect to the Lebesgue measure 95 l .

Theorem 2. Let $k > 1$. Then the distribution function of the random variable $\overline{D}_n^{(k)}(x)$ with respect to the Lebesgue measure is given as follows:

$$\Phi_n^{(k)}(t) = \begin{cases} 0, \quad если \quad t < q_n\, q_{n+1}^{-1}; \\ \sum\limits_{i=0}^{q_n-1} l(T^i(\Delta_0^{(n+1)} \cap \pi_n^{-k} \Delta_0^{(n+1)})) + \\ + \sum\limits_{j=0}^{q_{n+1}-1} l(T^j(\Delta_0^{(n)} \cap \pi_n^{-k} \Delta_0^{(n+1)})), \quad если \quad q_n q_{n+1}^{-1} \le t < 1; \\ 1, \quad если \quad t \ge 1. \end{cases}$$

In this paper, we formulate and prove a limit theorem for the sequence of time distribution functions of the $k-$th hit $\Phi_n^{(k)}(t),\ k > 1$.

Theorem 3. Let a homeomorphism $T \in B(T_b)$, and $\Phi_n^{(k)}(t)$ -be the distribution function of the random variable $\overline{D}_n^{(k)}(x)$. Then

1) for all $t \in R^1$ there is a finite limit of $\lim\limits_{n\to\infty} \Phi_n^{(k)}(t) = \Phi^{(k)}(t)$,

with $\Phi^{(k)}(t) = 0$ if $t \le 0$ and $\Phi^{(k)}(t) = 1$ if $t \ge 1$;

2) function $\Phi^{(k)}(t)$ is a step function on $[0, 1]$ with two points

gap.

Proof of Theorem 3. Suppose that $k > 1$. The distribution function of a random variable $\overline{D}_n^{(k)}(x)$ -is a step function that takes only three values. Therefore, we will prove the existence of the limit $\lim\limits_{n\to\infty} \overline{D}_n^{(k)}(t)$ in three steps.

1) $D_n^{(k)}(t) = 0$ if $t < q_n\, q_{n+1}^{-1}$. Considering

$$\lim_{n\to\infty} \frac{q_n}{q_{n+1}} = \lim_{n\to\infty} \frac{p_{n+1}}{q_{n+1}} = \rho$$

we get that $\lim\limits_{n\to\infty} \Phi_n^{(k)}(t) = 0$ if $t \le \rho$.

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2) We have

$$\lim_{n\to\infty} \Phi_n^{(k)}(t)=1, \text{ if если } t\ge 1.$$

3) Now we prove the existence of the limit of the sum

$$\sum_{i=0}^{q_n-1} l(T^i(\Delta_0^{(n+1)}\cap \pi_n^{-k}(\Delta_0^{(n+1)})))+\sum_{j=0}^{q_{n+1}-1} l(T^j(\Delta_0^{(n)}\cap \pi_n^{-k}(\Delta_0^{(n+1)}))) \quad (1)$$

First we have to figure out the structure of the $\pi_n^{-1}(\Delta_0^{(n+1)})\cap V_n$ set.

Let's write the explicit form of the function $\pi_n^{-1}(x)$:

$$\pi_n^{-1}(x)=\begin{cases} T^{-q_n}x, & если \quad x\in[x_{q_n}, x_{q_{n+2}}), \\ T^{-q_{n+1}}x, & если \quad x\in[x_{q_{n+2}}, x_{q_{n+1}}) \end{cases}$$

Function $\pi^{-1}(x)$, as can be seen from the last formula, has a gap only at point $x=x_{q_{n+2}}$. Therefore, for any interval $I\subset V_n$, region $\pi^{-1}(I)$ is an interval if $x_{q_{n+2}}\in I$, or the sum of two intervals if I does not contain point $x_{q_{n+2}}$, or the sum of two intervals if $x_{q_n}\in I$. Hence it follows that

$$\Delta_0^{(n+1)}\cap \pi_n^{-k}(\Delta_0^{(n+1)})=\bigcup_{m=1}^{l_1(k)}\omega'_m, \quad \Delta_0^{(n)}\cap \pi_n^{-k}(\Delta_0^{(n+1)})=\bigcup_{p=1}^{l_2(k)}\omega''_p$$

where ω'_m and ω''_p - are intervals such that $\omega'_m\subset\Delta_0^{(n+1)},\ 1\le m\le l_1(k);$ $\omega''_p\subset\Delta_0^{(n)},\ 1\le p\le l_2(k)$. Note that $l_1(k)+l_2(k)\le 2^k$. We denote the sum (1) by S_n and write it in the following form:

$$S_n=\sum_{m=1}^{l_1(k)}\sum_{i=0}^{q_n-1} l(T^i(\omega'_m))+\sum_{p=1}^{l_2(k)}\sum_{j=0}^{q_{n+1}-1} l(T^j(\omega''_p)) .$$

By virtue of the assertion of Theorem 1. [3], the sums $\sum_{i=0}^{q_n-1} l(T^i(\omega'_m))$ and $\sum_{j=0}^{q_{n+1}-1} l(T^j(\omega''_p))$ converge at $n\to\infty$, hence it follows that there is a finite limit of the sum S_n at $S_n\to\infty$. Theorem 3 is proved.

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